# COUNTING $(1, \beta)$-BM RELATIONS AND CLASSIFYING $(2, 2)$-BM GROUPS

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Abstract. In the first part, we prove that the number of $(1, \beta)$-BM relations is $3 \cdot 5 \cdot \ldots \cdot (2\beta + 1)$, which was conjectured by Jason Kimberley. In the second part, we construct two isomorphisms between certain $(2, 2)$-BM groups. This completes the classification of $(2, 2)$-BM groups initiated in [4].

## 1. Introduction

Let $\mathcal{T}_r$ be the $r$-regular tree and $\mathrm{Aut}(\mathcal{T}_r)$ its group of automorphisms. If $\alpha, \beta \in \mathbb{N}$, an $(\alpha, \beta)$-*BM group* is a torsion-free subgroup of $\mathrm{Aut}(\mathcal{T}_{2\alpha}) \times \mathrm{Aut}(\mathcal{T}_{2\beta})$ acting freely and transitively on the vertex set of the affine building $\mathcal{T}_{2\alpha} \times \mathcal{T}_{2\beta}$.

The class of $(\alpha, \beta)$-BM groups includes for example $F_\alpha \times F_\beta$ (the direct product of free groups of rank $\alpha$ and $\beta$), but also more complicated groups, like groups containing a finitely presented, torsion-free, simple subgroup of finite index, if $\alpha$ and $\beta$ are large enough, see [1, Theorem 6.4]. The first (and only known) examples of finitely presented, torsion-free, simple groups have been found in this way. See also [7, Section II.5] for a non-residually finite $(4, 3)$-BM group, [6, Example 2.3] for a $(3, 3)$-BM group having no non-trivial normal subgroups of infinite index, and [6, Example 3.4] for a $(6, 4)$-BM group having a subgroup of index 4 which is finitely presented, torsion-free, and simple.

An equivalent definition for an $(\alpha, \beta)$-BM group is the following (the equivalence is shown in [4, Theorem 3.4]): Let $A_\alpha = \{a_1, \ldots, a_\alpha\}$, $B_\beta = \{b_1, \ldots, b_\beta\}$, $a, a' \in A_\alpha^{\pm 1}$, and $b, b' \in B_\beta^{\pm 1}$. We think of the elements in $A_\alpha^{\pm 1}$ as oriented horizontal edges and the elements in $B_\beta^{\pm 1}$ as oriented vertical edges. A *geometric square* $[aba'b']$ is a set (consisting of a usual oriented square $aba'b'$ and reflections along its edges)

$$[aba'b'] := \{aba'b',\, a'b'ab,\, a^{-1}b'^{-1}a'^{-1}b^{-1},\, a'^{-1}b^{-1}a^{-1}b'^{-1}\}.$$

See Figure 1 for an illustration of the geometric square $[aba'b']$.

It is easy to check that

$$[aba'b'] = [a'b'ab] = [a^{-1}b'^{-1}a'^{-1}b^{-1}] = [a'^{-1}b^{-1}a^{-1}b'^{-1}].$$

Let $GS_{\alpha,\beta}$ be the set of all such geometric squares.

$$GS_{\alpha,\beta} := \{[aba'b'] : a, a' \in A_\alpha^{\pm 1},\, b, b' \in B_\beta^{\pm 1}\}.$$

Given a subset $S \subseteq GS_{\alpha,\beta}$, the *link* $Lk(S)$ is an undirected graph with vertex set $A_\alpha^{\pm 1} \sqcup B_\beta^{\pm 1}$ and edges $\{a^{-1}, b\}$, $\{a', b^{-1}\}$, $\{a'^{-1}, b'\}$, $\{a, b'^{-1}\}$ for each geometric square $[aba'b'] \in S$. These edges in the link correspond to the four corners in $aba'b'$. An $(\alpha, \beta)$-*BM relation* is a set $R$ consisting of exactly $\alpha\beta$ geometric squares in

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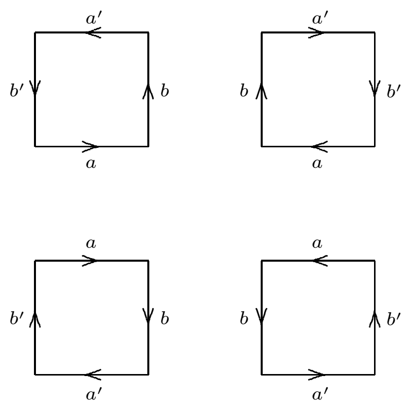


FIGURE 1. The geometric square $[aba'b']$, represented by each of these four squares.

$GS_{\alpha,\beta}$ such that $Lk(R)$ is the complete bipartite graph $K_{2\alpha,2\beta}$ (where the bipartite structure is induced by the decomposition $A_\alpha^{\pm 1} \sqcup B_\beta^{\pm 1}$). This *link condition* for $R$ means that for any given $a \in A_\alpha^{\pm 1}$, $b \in B_\beta^{\pm 1}$, there are unique $a' \in A_\alpha^{\pm 1}$, $b' \in B_\beta^{\pm 1}$ such that $[aba'b'] \in R$. It also excludes the existence of geometric squares of the form $[abab]$ in an $(\alpha,\beta)$-BM relation by a simple counting argument ($K_{2\alpha,2\beta}$ has $2\alpha + 2\beta$ vertices and $2\alpha \cdot 2\beta = 4\alpha\beta$ edges, so each of the $\alpha\beta$ geometric squares in $R$ has to contribute four distinct edges, but $[abab]$ only contributes the two edges $\{a^{-1}, b\}$ and $\{a, b^{-1}\}$). We denote by $R_{\alpha,\beta}$ the set of $(\alpha,\beta)$-BM relations. Any group $\Gamma$ with a finite presentation $\langle A_\alpha \cup B_\beta \mid R\rangle$, where $R \in R_{\alpha,\beta}$, is called an $(\alpha,\beta)$-*BM group*. Note that any of the four squares representing a geometric square induces the same relation in $\Gamma$, and that therefore any $(\alpha,\beta)$-BM group has a presentation with $\alpha+\beta$ generators and $\alpha\beta$ relations of the form $aba'b'$.

The cardinality of $R_{\alpha,\beta}$ (i.e. the number of $(\alpha,\beta)$-BM relations) has been computed for a finite number of small pairs $(\alpha,\beta)$ in [5, Table B.3] and independently with a different method in [3, Table 4], see Table 1.

In the smallest case, we have $|R_{1,1}| = 3$, since

$$R_{1,1} = \big\{\{[a_1b_1a_1^{-1}b_1^{-1}]\}, \{[a_1b_1a_1b_1^{-1}]\}, \{[a_1b_1a_1^{-1}b_1]\}\big\},$$

using the observation that

$$\{[a_1b_1a_1b_1]\} = \{[a_1^{-1}b_1^{-1}a_1^{-1}b_1^{-1}]\} \notin R_{1,1}$$

and

$$\{[a_1b_1^{-1}a_1b_1^{-1}]\} = \{[a_1^{-1}b_1a_1^{-1}b_1]\} \notin R_{1,1}.$$

In general, let's say if $\alpha\beta > 10$, the value $|R_{\alpha,\beta}|$ is not known, but Kimberley has conjectured in [3, Conjecture 193] that $|R_{1,\beta}| = 3 \cdot 5 \cdot \ldots \cdot (2\beta+1)$ for all $\beta \in \mathbb{N}$. We will prove this conjecture in Section 2. Observe that $|R_{\alpha,\beta}| = |R_{\beta,\alpha}|$ and therefore $|R_{\alpha,1}| = 3 \cdot 5 \cdot \ldots \cdot (2\alpha+1)$ for all $\alpha \in \mathbb{N}$.

Each element $R \in R_{\alpha,\beta}$ defines the $(\alpha,\beta)$-BM group $\langle A_\alpha \cup B_\beta \mid R\rangle$. Of course, it is possible that distinct $(\alpha,\beta)$-BM relations define isomorphic $(\alpha,\beta)$-BM group, for example (taking $\alpha = \beta = 1$)

$$\langle a_1, b_1 \mid a_1b_1a_1b_1^{-1}\rangle \cong \langle a_1, b_1 \mid a_1b_1a_1^{-1}b_1\rangle,$$

| $\alpha$ | $\beta$ | $\lvert R_{\alpha,\beta}\rvert$ | |
|---|---|---|---|
| 1 | 1 | 3 | |
| 1 | 2 | 15 | $= 3 \cdot 5$ |
| 1 | 3 | 105 | $= 3 \cdot 5 \cdot 7$ |
| 1 | 4 | 945 | $= 3 \cdot 5 \cdot 7 \cdot 9$ |
| 1 | 5 | 10395 | $= 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11$ |
| 1 | 6 | 135135 | $= 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13$ |
| 1 | 7 | 2027025 | $= 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15$ |
| 1 | 8 | 34459425 | $= 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17$ |
| 1 | 9 | 654729075 | $= 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot 19$ |
| 2 | 2 | 541 | prime |
| 2 | 3 | 35235 | $= 3^5 \cdot 5 \cdot 29$ |
| 2 | 4 | 3690009 | $= 3^3 \cdot 19 \cdot 7193$ |
| 2 | 5 | 570847095 | $= 3^6 \cdot 5 \cdot 7 \cdot 13 \cdot 1721$ |
| 3 | 3 | 27712191 | $= 3 \cdot 13 \cdot 710569$ |

TABLE 1. Number of $(\alpha,\beta)$-BM relations, $\alpha \le \beta$.

whereas $\{[a_1 b_1 a_1 b_1^{-1}]\} \neq \{[a_1 b_1 a_1^{-1} b_1]\}$. The classification of $(\alpha,\beta)$-BM groups up to isomorphism seems to be a hard problem in general (even if the set $R_{\alpha,\beta}$ is known). It has been done by Kimberley in [3, Chapter 5] for $(1,\beta)$-BM groups, if $\beta \in \{1,\dots,5\}$. Moreover, Kimberley and Robertson have proved that there are at least 41 and at most 43 $(2,2)$-BM groups up to isomorphism, see [4, Section 7] and [3, Chapter 5]. Starting from a reservoir of $|R_{2,2}| = 541$ $(2,2)$-BM relations, the lower bound was achieved by computing the abelianizations of the corresponding $(2,2)$-BM groups, and the abelianizations of subgroups of low index. The upper bound comes from constructing isomorphisms via generator permutations and Tietze transformations. It remained the open question whether the group $\Gamma_4$ is isomorphic to $\Gamma_{30}$ and whether $\Gamma_5$ is isomorphic to $\Gamma_{10}$ (these four $(2,2)$-BM groups will be defined in Section 3). We will give a positive answer by constructing explicit isomorphisms, such that there are in fact exactly 41 $(2,2)$-BM groups up to isomorphism. If $\alpha,\beta \ge 2$, no other complete classification of $(\alpha,\beta)$-BM groups is known so far.

## 2. Counting $(1,\beta)$-BM relations

In this section, we will define a map $\psi_\beta$ which associates to any $(1,\beta)$-BM relation $R \in R_{1,\beta}$ a set $\psi_\beta(R) = \psi_\beta^{(1)}(R) \cup \psi_\beta^{(2)}(R)$ consisting of $3+2\beta$ distinct $(1,\beta+1)$-BM relations (see Lemma 2 and Lemma 3). These $3+2\beta$ elements are either obtained by adding to $R$ a single new geometric square, or by first removing from $R$ one of the $\beta$ geometric squares and then adding two suitably chosen new geometric squares. Distinct elements $R$, $T$ in $R_{1,\beta}$ will produce disjoint sets $\psi_\beta(R)$, $\psi_\beta(T)$ (see Lemma 4). Moreover, any $(1,\beta+1)$-BM relation can be obtained by $\psi_\beta$ (see Lemma 5). This allows us to compute inductively the exact number of $(1,\beta)$-BM relations for any $\beta \in \mathbb{N}$, and therefore to prove Kimberley's conjecture.

Let $R = \{r_1, \dots, r_\beta\} \in R_{1,\beta}$, i.e. $r_1, \dots, r_\beta$ are $\beta$ geometric squares in $GS_{1,\beta}$ satisfying the link condition $Lk(\{r_1, \dots, r_\beta\}) = K_{2,2\beta}$. We first define

$$\begin{aligned}\psi_\beta^{(1)}(R) := \big\{&\{r_1, \dots, r_\beta, [a_1 b_{\beta+1} a_1^{-1} b_{\beta+1}^{-1}]\},\\ &\{r_1, \dots, r_\beta, [a_1 b_{\beta+1} a_1 b_{\beta+1}^{-1}]\},\\ &\{r_1, \dots, r_\beta, [a_1 b_{\beta+1} a_1^{-1} b_{\beta+1}]\}\big\},\end{aligned}$$

a set consisting of three distinct $(1, \beta+1)$-BM relations.

If $[aba'b'] \in GS_{1,\beta}$, we define

$$\phi_\beta([aba'b']) := \big\{\{[ab_{\beta+1}a'b'], [aba'b_{\beta+1}^{-1}]\}, \{[ab_{\beta+1}^{-1}a'b'], [aba'b_{\beta+1}]\}\big\}.$$

**Lemma 1.** *The map $\phi_\beta$ is well-defined.*

*Proof.* We have to show

$$\phi_\beta([aba'b']) = \phi_\beta([a'b'ab]) = \phi_\beta([a^{-1}b'^{-1}a'^{-1}b^{-1}]) = \phi_\beta([a'^{-1}b^{-1}a^{-1}b'^{-1}]).$$

Let $v_1 := [ab_{\beta+1}a'b']$, $v_2 := [aba'b_{\beta+1}^{-1}]$, $v_3 := [ab_{\beta+1}^{-1}a'b']$ and $v_4 := [aba'b_{\beta+1}]$, such that we have $\{v_1, v_2, v_3, v_4\} \subset GS_{1,\beta+1}$ and $\phi_\beta([aba'b']) = \big\{\{v_1, v_2\}, \{v_3, v_4\}\big\}$.

We check that

$$\begin{aligned}\phi_\beta([a'b'ab]) &= \big\{\{[a'b_{\beta+1}ab], [a'b'ab_{\beta+1}^{-1}]\}, \{[a'b_{\beta+1}^{-1}ab], [a'b'ab_{\beta+1}]\}\big\}\\ &= \big\{\{v_4, v_3\}, \{v_2, v_1\}\big\} = \phi_\beta([aba'b']),\end{aligned}$$

$$\begin{aligned}\phi_\beta([a^{-1}b'^{-1}a'^{-1}b^{-1}]) &= \big\{\{[a^{-1}b_{\beta+1}a'^{-1}b^{-1}], [a^{-1}b'^{-1}a'^{-1}b_{\beta+1}^{-1}]\},\\ &\qquad \{[a^{-1}b_{\beta+1}^{-1}a'^{-1}b^{-1}], [a^{-1}b'^{-1}a'^{-1}b_{\beta+1}]\}\big\}\\ &= \big\{\{v_2, v_1\}, \{v_4, v_3\}\big\} = \phi_\beta([aba'b']),\end{aligned}$$

$$\begin{aligned}\phi_\beta([a'^{-1}b^{-1}a^{-1}b'^{-1}]) &= \big\{\{[a'^{-1}b_{\beta+1}a^{-1}b'^{-1}], [a'^{-1}b^{-1}a^{-1}b_{\beta+1}^{-1}]\},\\ &\qquad \{[a'^{-1}b_{\beta+1}^{-1}a^{-1}b'^{-1}], [a'^{-1}b^{-1}a^{-1}b_{\beta+1}]\}\big\}\\ &= \big\{\{v_3, v_4\}, \{v_1, v_2\}\big\} = \phi_\beta([aba'b']).\end{aligned}$$

□

See Figure 2 for a visualization of the map $\phi_\beta$.

We now construct the set $\psi_\beta^{(2)}(R)$ consisting of $2\beta$ distinct $(1, \beta+1)$-BM relations (as we will prove later). Let

$$\psi_\beta^{(2)}(R) := \bigcup_{i=1}^{\beta} \left( \bigcup_{P \in \phi_\beta(r_i)} \big\{P \cup (R \setminus \{r_i\})\big\} \right).$$

Note that if $r_i = [aba'b']$ then by definition of $\phi_\beta$

$$\bigcup_{P \in \phi_\beta(r_i)} \big\{P \cup (R \setminus \{r_i\})\big\} =$$

$$\big\{\{[ab_{\beta+1}a'b'], [aba'b_{\beta+1}^{-1}]\} \cup (R \setminus \{r_i\}),\ \{[ab_{\beta+1}^{-1}a'b'], [aba'b_{\beta+1}]\} \cup (R \setminus \{r_i\})\big\}.$$

Finally, let

$$\psi_\beta(R) := \psi_\beta^{(1)}(R) \cup \psi_\beta^{(2)}(R).$$

See Section 4 for an explicit construction of the map $\psi_\beta$ in the case $\beta = 1$ and $\beta = 2$.

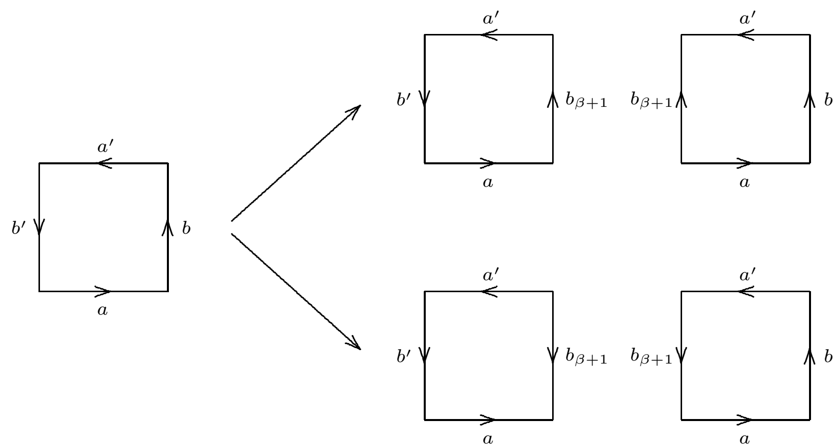


FIGURE 2. The map $\phi_\beta$ associates to the geometric square $[aba'b'] \in GS_{1,\beta}$ (represented on the left) the two geometric squares in $GS_{1,\beta+1}$ represented on top right, and the two geometric squares in $GS_{1,\beta+1}$ represented on bottom right, respectively.

**Lemma 2.** *If $R \in R_{1,\beta}$, then the elements in $\psi_\beta(R)$ are $(1,\beta+1)$-BM relations.*

*Proof.* The statement is clear for the three elements in $\psi_\beta^{(1)}(R)$ looking at their link.

To show it for the elements in $\psi_\beta^{(2)}(R)$, first note that

$$[ab_{\beta+1}a'b'] \neq [aba'b_{\beta+1}^{-1}] \; (= [a^{-1}b_{\beta+1}a'^{-1}b^{-1}])$$

and

$$[ab_{\beta+1}^{-1}a'b'] \neq [aba'b_{\beta+1}] \; (= [a^{-1}b_{\beta+1}^{-1}a'^{-1}b^{-1}]).$$

Therefore each element in $\psi_\beta^{(2)}(R)$ consists of $\beta+1$ geometric squares in $GS_{1,\beta+1}$. Let $R = \{r_1, \dots, r_\beta\} \in R_{1,\beta}$, fix any $i \in \{1, \dots, \beta\}$, and suppose that $r_i = [aba'b']$. Since $Lk(R) = K_{2,2\beta}$, we have

$$Lk(\{r_1, \dots, r_\beta, [ab_{\beta+1}a'b_{\beta+1}^{-1}]\}) = K_{2,2\beta+2},$$

independently of $a, a' \in \{a_1, a_1^{-1}\} = A_1^{\pm 1}$. Since $r_i = [aba'b']$, we can write this as

$$K_{2,2\beta+2} = Lk(\{[aba'b'], [ab_{\beta+1}a'b_{\beta+1}^{-1}]\} \cup (R \setminus \{r_i\})),$$

which can directly be seen to be equal to

$$Lk(\{[ab_{\beta+1}a'b'], [aba'b_{\beta+1}^{-1}]\} \cup (R \setminus \{r_i\})),$$

since the edges in $Lk(\{[aba'b'], [ab_{\beta+1}a'b_{\beta+1}^{-1}]\})$ are

$$\{a^{-1}, b\}, \{a', b^{-1}\}, \{a'^{-1}, b'\}, \{a, b'^{-1}\},$$
$$\{a^{-1}, b_{\beta+1}\}, \{a', b_{\beta+1}^{-1}\}, \{a'^{-1}, b_{\beta+1}^{-1}\}, \{a, b_{\beta+1}\},$$

which are also the edges in $Lk(\{[ab_{\beta+1}a'b'], [aba'b_{\beta+1}^{-1}]\})$. In fact, we have performed a link preserving surgery as described more generally in [1, Section 6.2.2].

Similarly (interchanging $b_{\beta+1}$ and $b_{\beta+1}^{-1}$) one proves that

$$Lk(\{[ab_{\beta+1}^{-1}a'b'],[aba'b_{\beta+1}]\}\cup(R\setminus\{r_i\}))=K_{2,2\beta+2}.$$

□

**Lemma 3.** *If $R\in R_{1,\beta}$, then $|\psi_\beta(R)|=3+2\beta$.*

*Proof.* Clearly $|\psi_\beta^{(1)}(R)|=3$.

The label $b_{\beta+1}$ or $b_{\beta+1}^{-1}$ appears in exactly one geometric square of each element in $\psi_\beta^{(1)}(R)$, but in exactly two geometric squares of each element in $\psi_\beta^{(2)}(R)$, hence we conclude

$$\psi_\beta^{(1)}(R)\cap\psi_\beta^{(2)}(R)=\emptyset.$$

Let $R=\{r_1,\dots,r_\beta\}$. Fix any $i\in\{1,\dots,\beta\}$ and suppose that $r_i=[aba'b']$. The geometric square $r_i$ only misses in the two elements

$$\{[ab_{\beta+1}a'b'],[aba'b_{\beta+1}^{-1}]\}\cup(R\setminus\{r_i\})$$

and

$$\{[ab_{\beta+1}^{-1}a'b'],[aba'b_{\beta+1}]\}\cup(R\setminus\{r_i\})$$

of $\psi_\beta^{(2)}(R)$. Suppose that they are equal. Then

$$\{[ab_{\beta+1}a'b'],[aba'b_{\beta+1}^{-1}]\}=\{[ab_{\beta+1}^{-1}a'b'],[aba'b_{\beta+1}]\}.$$

It follows that

$$[ab_{\beta+1}a'b']=[aba'b_{\beta+1}]\ (=[a'b_{\beta+1}ab]),$$

since $[ab_{\beta+1}a'b']\neq[ab_{\beta+1}^{-1}a'b']$, but then $a=a'$ and $b=b'$. This is impossible, since $[abab]\notin R\in R_{1,\beta}$. This shows that the $2\beta$ elements in $\psi_\beta^{(2)}(R)$ are distinct, and we get

$$|\psi_\beta(R)|=|\psi_\beta^{(1)}(R)\cup\psi_\beta^{(2)}(R)|=|\psi_\beta^{(1)}(R)|+|\psi_\beta^{(2)}(R)|-|\psi_\beta^{(1)}(R)\cap\psi_\beta^{(2)}(R)|=3+2\beta.$$

□

**Lemma 4.** *If $R,T\in R_{1,\beta}$ and $R\neq T$, then $\psi_\beta(R)\cap\psi_\beta(T)=\emptyset$.*

*Proof.* Let $R=\{r_1,\dots,r_\beta\}$ and $T=\{t_1,\dots,t_\beta\}$. We suppose without loss of generality that $r_1=[aba'b']\notin T$. Then $r_1$ appears in no element of $\psi_\beta(T)$, but appears in each element of $\psi_\beta(R)$ except in

$$U_1:=\{[ab_{\beta+1}a'b'],[aba'b_{\beta+1}^{-1}]\}\cup(R\setminus\{r_1\})\in\psi_\beta^{(2)}(R)$$

and

$$V_1:=\{[ab_{\beta+1}^{-1}a'b'],[aba'b_{\beta+1}]\}\cup(R\setminus\{r_1\})\in\psi_\beta^{(2)}(R).$$

We want to show by contradiction that $U_1,V_1\notin\psi_\beta(T)$. It is clear that $U_1,V_1\notin\psi_\beta^{(1)}(T)$. Fix any $i\in\{1,\dots,\beta\}$ and let $t_i=[\breve{a}\breve{b}\hat{a}\hat{b}]$, where $\breve{a},\hat{a}\in\{a_1,a_1^{-1}\}$ and $\breve{b},\hat{b}\in B_\beta^{\pm1}$. We suppose that $U_1\in\psi_\beta^{(2)}(T)$ or $V_1\in\psi_\beta^{(2)}(T)$ and have therefore to consider four cases:

Case 1: Suppose that

$$U_1=\{[\breve{a}b_{\beta+1}\hat{a}\hat{b}],[\breve{a}\breve{b}\hat{a}b_{\beta+1}^{-1}]\}\cup(T\setminus\{t_i\}).$$

Then $R\setminus\{r_1\}=T\setminus\{t_i\}$ and

$$\{[ab_{\beta+1}a'b'],[aba'b_{\beta+1}^{-1}]\}=\{[\breve{a}b_{\beta+1}\hat{a}\hat{b}],[\breve{a}\breve{b}\hat{a}b_{\beta+1}^{-1}]\}.$$

Case 1.1: If $[ab_{\beta+1}a'b'] = [\breve{a}b_{\beta+1}\hat{a}\hat{b}]$ and $[aba'b_{\beta+1}^{-1}] = [\breve{a}\breve{b}\hat{a}b_{\beta+1}^{-1}]$, then $a = \breve{a}$, $b = \breve{b}$, $a' = \hat{a}$ and $b' = \hat{b}$. This implies $t_i = [aba'b'] = r_1$, hence $R = T$, a contradiction.

Case 1.2: If

$$[ab_{\beta+1}a'b'] = [\breve{a}\breve{b}\hat{a}b_{\beta+1}^{-1}] \ (= [\breve{a}^{-1}b_{\beta+1}\hat{a}^{-1}\breve{b}^{-1}])$$

and

$$[aba'b_{\beta+1}^{-1}] = [\breve{a}b_{\beta+1}\hat{a}\hat{b}] \ (= [\breve{a}^{-1}\hat{b}^{-1}\hat{a}^{-1}b_{\beta+1}^{-1}]),$$

then $a = \breve{a}^{-1}$, $b = \hat{b}^{-1}$, $a' = \hat{a}^{-1}$ and $b' = \breve{b}^{-1}$. This implies

$$t_i = [a^{-1}b'^{-1}a'^{-1}b^{-1}] = [aba'b'] = r_1$$

and again the contradiction $R = T$.

The three remaining cases

Case 2: $U_1 = \{[\breve{a}b_{\beta+1}^{-1}\hat{a}\hat{b}], [\breve{a}\breve{b}\hat{a}b_{\beta+1}]\} \cup (T \setminus \{t_i\})$

Case 3: $V_1 = \{[\breve{a}b_{\beta+1}\hat{a}\hat{b}], [\breve{a}\breve{b}\hat{a}b_{\beta+1}^{-1}]\} \cup (T \setminus \{t_i\})$

Case 4: $V_1 = \{[\breve{a}b_{\beta+1}^{-1}\hat{a}\hat{b}], [\breve{a}\breve{b}\hat{a}b_{\beta+1}]\} \cup (T \setminus \{t_i\})$

can be treated similarly. In fact we can reduce them to Case 1 as follows:

In Case 2, since

$$\{[\breve{a}b_{\beta+1}^{-1}\hat{a}\hat{b}], [\breve{a}\breve{b}\hat{a}b_{\beta+1}]\} = \{[\hat{a}^{-1}b_{\beta+1}\breve{a}^{-1}\hat{b}^{-1}], [\hat{a}^{-1}\breve{b}^{-1}\breve{a}^{-1}b_{\beta+1}^{-1}]\},$$

we can substitute $\breve{a}\breve{b}\hat{a}\hat{b}$ by $\hat{a}^{-1}\breve{b}^{-1}\breve{a}^{-1}\hat{b}^{-1}$ and are in Case 1.

In Case 3 and Case 4, since

$$(V_1 \cup \{r_1\}) \setminus R = \{[ab_{\beta+1}^{-1}a'b'], [aba'b_{\beta+1}]\} = \{[a'^{-1}b_{\beta+1}a^{-1}b'^{-1}], [a'^{-1}b^{-1}a^{-1}b_{\beta+1}^{-1}]\},$$

we can substitute $aba'b'$ by $a'^{-1}b^{-1}a^{-1}b'^{-1}$ and are in Case 1 and Case 2, respectively.

Thus, we have shown that the only two elements $U_1, V_1$ of $\psi_\beta(R)$ in which $r_1$ does not appear, are no elements of $\psi_\beta(T)$, and therefore $\psi_\beta(R) \cap \psi_\beta(T) = \emptyset$. □

**Lemma 5.** *Let $U \in R_{1,\beta+1}$. Then $U \in \psi_\beta(R)$ for some $R \in R_{1,\beta}$.*

*Proof.* Let $U = \{u_1, \ldots, u_{\beta+1}\} \in R_{1,\beta+1}$. By the link condition, the label $b_{\beta+1}$ or $b_{\beta+1}^{-1}$ appears either in exactly one or in exactly two elements (geometric squares) of $U$.

Case 1: Suppose that the label $b_{\beta+1}$ or $b_{\beta+1}^{-1}$ appears in exactly one element of $U$, say in $u_{\beta+1}$. Then either

$$u_{\beta+1} = [a_1 b_{\beta+1} a_1^{-1} b_{\beta+1}^{-1}]$$

or

$$u_{\beta+1} = [a_1 b_{\beta+1} a_1 b_{\beta+1}^{-1}]$$

or

$$u_{\beta+1} = [a_1 b_{\beta+1} a_1^{-1} b_{\beta+1}].$$

Let $R := \{u_1, \ldots, u_\beta\} = U \setminus \{u_{\beta+1}\}$. Note that $R \in R_{1,\beta}$, since $Lk(U) = K_{2,2\beta+2}$ and $u_{\beta+1}$ contributes to $Lk(U)$ the four edges

$$\{a_1^{-1}, b_{\beta+1}\}, \{a_1^{-1}, b_{\beta+1}^{-1}\}, \{a_1, b_{\beta+1}^{-1}\}, \{a_1, b_{\beta+1}\},$$

independently of the three possibilities for $u_{\beta+1}$. By definition of $\psi_\beta^{(1)}$ we have $U \in \psi_\beta^{(1)}(R) \subset \psi_\beta(R)$.

Case 2: Suppose that the label $b_{\beta+1}$ or $b_{\beta+1}^{-1}$ appears in $u_\beta$ and $u_{\beta+1}$, but in no other element of $U$. It follows that $u_\beta = [ab_{\beta+1}a'b']$ for some $a, a' \in \{a_1, a_1^{-1}\}$ and $b' \in B_\beta^{\pm 1}$. In particular $b' \neq b_{\beta+1}$ and $b' \neq b_{\beta+1}^{-1}$, otherwise we would be in Case 1. Looking at the link of $\{u_1, \ldots, u_\beta\} = U \setminus \{u_{\beta+1}\}$, we see that the two edges $\{a^{-1}, b_{\beta+1}\}$ and $\{a', b_{\beta+1}^{-1}\}$ in this link (and two other edges not involving the label $b_{\beta+1}$ or $b_{\beta+1}^{-1}$) are contributed by $u_\beta$. The edges contributed by $\{u_1, \ldots, u_{\beta-1}\}$ do not involve $b_{\beta+1}$ or $b_{\beta+1}^{-1}$. Therefore, the two edges $\{a, b_{\beta+1}\}$ and $\{a'^{-1}, b_{\beta+1}^{-1}\}$ (and two other edges not involving the label $b_{\beta+1}$ or $b_{\beta+1}^{-1}$) are missing to get the complete bipartite graph $K_{2,2\beta+2} = Lk(U)$. Hence $u_{\beta+1} = [aba'b_{\beta+1}^{-1}]$ for some $b \in B_\beta^{\pm 1}$. Let $R := \{u_1, \ldots, u_{\beta-1}, [aba'b']\}$. Then $R \in R_{1,\beta}$ (i.e. $Lk(R) = K_{2,2\beta}$), since

$$\begin{aligned}
K_{2,2\beta+2} = Lk(U) &= Lk(\{u_1, \ldots, u_{\beta-1}, [ab_{\beta+1}a'b'], [aba'b_{\beta+1}^{-1}]\}) \\
&= Lk(\{u_1, \ldots, u_{\beta-1}, [aba'b'], [ab_{\beta+1}a'b_{\beta+1}^{-1}]\}) \\
&= Lk(R \cup \{[ab_{\beta+1}a'b_{\beta+1}^{-1}]\}).
\end{aligned}$$

By construction of $R$ and the definition of $\psi_\beta^{(2)}$, we have $U \in \psi_\beta^{(2)}(R) \subset \psi_\beta(R)$. $\square$

**Corollary 6.** *For $\beta \in \mathbb{N}$ we have*

$$\bigcup_{R \in R_{1,\beta}} \psi_\beta(R) = R_{1,\beta+1},$$

*in particular the set $R_{1,\beta+1}$ can be explicitly constructed from $R_{1,\beta}$ using $\psi_\beta$.*

*Proof.* Lemma 2 shows that

$$\bigcup_{R \in R_{1,\beta}} \psi_\beta(R) \subseteq R_{1,\beta+1}.$$

Moreover, we have

$$\bigcup_{R \in R_{1,\beta}} \psi_\beta(R) \supseteq R_{1,\beta+1}$$

by Lemma 5. $\square$

Note that the union in Corollary 6 is a *disjoint* union by Lemma 4. Now, we are able to prove Kimberley's conjecture on the number of $(1, \beta)$-BM relations.

**Theorem 7.** *([3, Conjecture 193]) For every positive integer $\beta$, the number of $(1, \beta)$-BM relations is*

$$|R_{1,\beta}| = \prod_{i=1}^{\beta} (2i+1).$$

*Proof.* By Lemma 3 and Lemma 5

$$|R_{1,\beta+1}| \leq (3 + 2\beta)|R_{1,\beta}|.$$

By Lemma 3 and Lemma 4

$$|R_{1,\beta+1}| \geq (3 + 2\beta)|R_{1,\beta}|,$$

hence

$$|R_{1,\beta+1}| = (3 + 2\beta)|R_{1,\beta}|.$$

The proof of the theorem is now by induction on $\beta$. If $\beta = 1$, then
$$R_{1,1} = \big\{\{[a_1b_1a_1^{-1}b_1^{-1}]\}, \{[a_1b_1a_1b_1^{-1}]\}, \{[a_1b_1a_1^{-1}b_1]\}\big\}$$
and $|R_{1,1}| = 3$. Assume that the statement of the theorem holds for $\beta$. Then
$$|R_{1,\beta+1}| = (3+2\beta)|R_{1,\beta}| = (2(\beta+1)+1)\prod_{i=1}^{\beta}(2i+1) = \prod_{i=1}^{\beta+1}(2i+1).$$
□

## 3. Classification of $(2,2)$-BM groups

Let $\Gamma_4$, $\Gamma_{30}$, $\Gamma_5$, $\Gamma_{10}$ be the $(2,2)$-BM groups
$$\Gamma_4 = \langle a,b,c,d \mid acac^{-1},\ adad^{-1},\ bcbd,\ bc^{-1}bd^{-1}\rangle,$$
$$\Gamma_{30} = \langle a,b,c,d \mid acad,\ ac^{-1}ad^{-1},\ bcbd,\ bc^{-1}bd^{-1}\rangle,$$
$$\Gamma_5 = \langle a,b,c,d \mid acac^{-1},\ adad^{-1},\ bcb^{-1}c,\ bdb^{-1}d\rangle,$$
$$\Gamma_{10} = \langle a,b,c,d \mid acac^{-1},\ ada^{-1}d,\ bcbc^{-1},\ bdb^{-1}d^{-1}\rangle.$$
(To simplify the notation, we use here the letters $a,b,c,d$ instead of $a_1,a_2,b_1,b_2$.)

We will prove that $\Gamma_4$ is isomorphic to $\Gamma_{30}$, and that $\Gamma_5$ is isomorphic to $\Gamma_{10}$. To find these isomorphisms we have written a program with GAP ([2]) using the normal form program developed in [5, Chapter B.6] and the knowledge of the orders of elements in the abelianizations of the four groups.

**Proposition 8.** *The groups $\Gamma_4$ and $\Gamma_{30}$ are isomorphic.*

*Proof.* Let $\eta : \Gamma_4 \to \Gamma_{30}$ be the homomorphism given by $\eta(a) = ab$, $\eta(b) = a$, $\eta(c) = ac$ and $\eta(d) = da^{-1}$. It is a homomorphism since
$$\eta(acac^{-1}) = abacabc^{-1}a^{-1} = abaa^{-1}d^{-1}bad = abd^{-1}bad = acad = 1,$$
$$\eta(adad^{-1}) = abda^{-1}abad^{-1} = abdbad^{-1} = abb^{-1}c^{-1}ad^{-1} = ac^{-1}ad^{-1} = 1,$$
$$\eta(bcbd) = aacada^{-1} = aaa^{-1}d^{-1}da^{-1} = 1,$$
$$\eta(bc^{-1}bd^{-1}) = ac^{-1}a^{-1}aad^{-1} = ac^{-1}ad^{-1} = 1,$$
using the four defining relations of $\Gamma_{30}$.

$\eta$ is surjective: $a = \eta(b)$, $b = \eta(b^{-1}a)$, $c = \eta(b^{-1}c)$, $d = \eta(db)$.

Let $\theta : \Gamma_{30} \to \Gamma_4$ be the homomorphism given by $\theta(a) = b$, $\theta(b) = b^{-1}a$, $\theta(c) = b^{-1}c$ and $\theta(d) = db$. It is a homomorphism since
$$\theta(acad) = bb^{-1}cbdb = cbdb = 1,$$
$$\theta(ac^{-1}ad^{-1}) = bc^{-1}bbb^{-1}d^{-1} = bc^{-1}bd^{-1} = 1,$$
$$\theta(bcbd) = b^{-1}ab^{-1}cb^{-1}adb = b^{-1}ab^{-1}bd^{-1}da^{-1}b = 1,$$
$$\theta(bc^{-1}bd^{-1}) = b^{-1}ac^{-1}bb^{-1}ab^{-1}d^{-1} = b^{-1}c^{-1}a^{-1}ab^{-1}d^{-1} = b^{-1}c^{-1}b^{-1}d^{-1} = 1,$$
using the four defining relations of $\Gamma_4$.

The composition $\theta \circ \eta$ is the identity on $\Gamma_4$, since
$$\theta(\eta(a)) = \theta(ab) = bb^{-1}a = a,$$
$$\theta(\eta(b)) = \theta(a) = b,$$
$$\theta(\eta(c)) = \theta(ac) = bb^{-1}c = c,$$
$$\theta(\eta(d)) = \theta(da^{-1}) = dbb^{-1} = d,$$
hence $\eta$ is injective and an isomorphism. □

**Proposition 9.** *The groups $\Gamma_5$ and $\Gamma_{10}$ are isomorphic.*

*Proof.* As in the proof of Proposition 8, it is easy to show that $\varphi : \Gamma_5 \to \Gamma_{10}$ defined by $\varphi(a) = d$, $\varphi(b) = ac$, $\varphi(c) = a$, $\varphi(d) = ab$ is an isomorphism. □

**Corollary 10.** *There are exactly* 41 $(2,2)$-*BM groups up to isomorphism.*

*Proof.* By [3, Proposition 222] there are at least 41 isomorphism classes of $(2,2)$-BM groups. By [3, Proposition 231] there are at most 43 isomorphism classes of $(2,2)$-BM groups (including the isomorphism classes of $\Gamma_4$, $\Gamma_{30}$, $\Gamma_5$ and $\Gamma_{10}$). Now use Proposition 8 and Proposition 9 to reduce the number of isomorphism classes from 43 to 41. □

## 4. Appendix: Illustration of $\psi_\beta$ for $\beta = 1$ and $\beta = 2$

In this appendix we first use the map $\psi_1$ to determine

$$\bigcup_{R \in R_{1,1}} \psi_1(R) = R_{1,2}.$$

Recall that

$$R_{1,1} = \big\{\{[a_1 b_1 a_1^{-1} b_1^{-1}]\}, \{[a_1 b_1 a_1 b_1^{-1}]\}, \{[a_1 b_1 a_1^{-1} b_1]\}\big\}.$$

By definition of $\psi_1^{(1)}$ and $\psi_1^{(2)}$, we have

$$\begin{aligned}
\psi_1^{(1)}(\{[a_1 b_1 a_1^{-1} b_1^{-1}]\}) = \big\{ & \{[a_1 b_1 a_1^{-1} b_1^{-1}], [a_1 b_2 a_1^{-1} b_2^{-1}]\}, \\
& \{[a_1 b_1 a_1^{-1} b_1^{-1}], [a_1 b_2 a_1 b_2^{-1}]\}, \\
& \{[a_1 b_1 a_1^{-1} b_1^{-1}], [a_1 b_2 a_1^{-1} b_2]\}\big\}. \\
\psi_1^{(2)}(\{[a_1 b_1 a_1^{-1} b_1^{-1}]\}) = \big\{ & \{[a_1 b_2 a_1^{-1} b_1^{-1}], [a_1 b_1 a_1^{-1} b_2^{-1}]\}, \\
& \{[a_1 b_2^{-1} a_1^{-1} b_1^{-1}], [a_1 b_1 a_1^{-1} b_2]\}\big\}.
\end{aligned}$$

$$\begin{aligned}
\psi_1^{(1)}(\{[a_1 b_1 a_1 b_1^{-1}]\}) = \big\{ & \{[a_1 b_1 a_1 b_1^{-1}], [a_1 b_2 a_1^{-1} b_2^{-1}]\}, \\
& \{[a_1 b_1 a_1 b_1^{-1}], [a_1 b_2 a_1 b_2^{-1}]\}, \\
& \{[a_1 b_1 a_1 b_1^{-1}], [a_1 b_2 a_1^{-1} b_2]\}\big\}. \\
\psi_1^{(2)}(\{[a_1 b_1 a_1 b_1^{-1}]\}) = \big\{ & \{[a_1 b_2 a_1 b_1^{-1}], [a_1 b_1 a_1 b_2^{-1}]\}, \\
& \{[a_1 b_2^{-1} a_1 b_1^{-1}], [a_1 b_1 a_1 b_2]\}\big\}.
\end{aligned}$$

$$\begin{aligned}
\psi_1^{(1)}(\{[a_1 b_1 a_1^{-1} b_1]\}) = \big\{ & \{[a_1 b_1 a_1^{-1} b_1], [a_1 b_2 a_1^{-1} b_2^{-1}]\}, \\
& \{[a_1 b_1 a_1^{-1} b_1], [a_1 b_2 a_1 b_2^{-1}]\}, \\
& \{[a_1 b_1 a_1^{-1} b_1], [a_1 b_2 a_1^{-1} b_2]\}\big\}. \\
\psi_1^{(2)}(\{[a_1 b_1 a_1^{-1} b_1]\}) = \big\{ & \{[a_1 b_2 a_1^{-1} b_1], [a_1 b_1 a_1^{-1} b_2^{-1}]\}, \\
& \{[a_1 b_2^{-1} a_1^{-1} b_1], [a_1 b_1 a_1^{-1} b_2]\}\big\}.
\end{aligned}$$

Taking the union of these six sets, we therefore obtain

$$\begin{aligned}R_{1,2} = \big\{&\{[a_1b_1a_1^{-1}b_1^{-1}],[a_1b_2a_1^{-1}b_2^{-1}]\},\ \{[a_1b_1a_1^{-1}b_1^{-1}],[a_1b_2a_1b_2^{-1}]\},\\
&\{[a_1b_1a_1^{-1}b_1^{-1}],[a_1b_2a_1^{-1}b_2]\},\ \{[a_1b_2a_1^{-1}b_1^{-1}],[a_1b_1a_1^{-1}b_2^{-1}]\},\\
&\{[a_1b_2^{-1}a_1^{-1}b_1^{-1}],[a_1b_1a_1^{-1}b_2]\},\ \{[a_1b_1a_1b_1^{-1}],[a_1b_2a_1^{-1}b_2^{-1}]\},\\
&\{[a_1b_1a_1b_1^{-1}],[a_1b_2a_1b_2^{-1}]\},\ \{[a_1b_1a_1b_1^{-1}],[a_1b_2a_1^{-1}b_2]\},\\
&\{[a_1b_2a_1b_1^{-1}],[a_1b_1a_1b_2^{-1}]\},\ \{[a_1b_2^{-1}a_1b_1^{-1}],[a_1b_1a_1b_2]\},\\
&\{[a_1b_1a_1^{-1}b_1],[a_1b_2a_1^{-1}b_2^{-1}]\},\ \{[a_1b_1a_1^{-1}b_1],[a_1b_2a_1b_2^{-1}]\},\\
&\{[a_1b_1a_1^{-1}b_1],[a_1b_2a_1^{-1}b_2]\},\ \{[a_1b_2a_1^{-1}b_1],[a_1b_1a_1^{-1}b_2^{-1}]\},\\
&\{[a_1b_2^{-1}a_1^{-1}b_1],[a_1b_1a_1^{-1}b_2]\}\big\}\end{aligned}$$

and $|R_{1,2}| = 15$. These 15 $(1,2)$-BM relations are also listed in [3, Table 7].

To illustrate what happens in the case $\beta = 2$, we take for example

$$R := \{[a_1b_1a_1^{-1}b_1^{-1}],[a_1b_2a_1^{-1}b_2^{-1}]\} \in R_{1,2},$$

and get seven $(1,3)$-BM relations

$$\begin{aligned}\psi_2^{(1)}(R) = \big\{&\{[a_1b_1a_1^{-1}b_1^{-1}],[a_1b_2a_1^{-1}b_2^{-1}],[a_1b_3a_1^{-1}b_3^{-1}]\},\\
&\{[a_1b_1a_1^{-1}b_1^{-1}],[a_1b_2a_1^{-1}b_2^{-1}],[a_1b_3a_1b_3^{-1}]\},\\
&\{[a_1b_1a_1^{-1}b_1^{-1}],[a_1b_2a_1^{-1}b_2^{-1}],[a_1b_3a_1^{-1}b_3]\}\big\}.\\
\psi_2^{(2)}(R) = \big\{&\{[a_1b_3a_1^{-1}b_1^{-1}],[a_1b_1a_1^{-1}b_3^{-1}],[a_1b_2a_1^{-1}b_2^{-1}]\},\\
&\{[a_1b_3^{-1}a_1^{-1}b_1^{-1}],[a_1b_1a_1^{-1}b_3],[a_1b_2a_1^{-1}b_2^{-1}]\},\\
&\{[a_1b_3a_1^{-1}b_2^{-1}],[a_1b_2a_1^{-1}b_3^{-1}],[a_1b_1a_1^{-1}b_1^{-1}]\},\\
&\{[a_1b_3^{-1}a_1^{-1}b_2^{-1}],[a_1b_2a_1^{-1}b_3],[a_1b_1a_1^{-1}b_1^{-1}]\}\big\}.\end{aligned}$$

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